

February 2, 1871.

General Sir EDWARD SABINE, K.C.B., President, in the Chair.

The following communications were read :—

I. "On Linear Differential Equations.—No. IV." By W. H. L. RUSSELL, F.R.S. Received November 17, 1870.

I will now consider some interesting results in definite integrals obtained from the solution of linear differential equations.

Let us first consider the linear differential equation

$$(a_3 + b_3 x) \frac{d^3 y}{dx^3} + (a_2 + b_2 x) \frac{d^2 y}{dx^2} + (a_1 + b_1 x) \frac{dy}{dx} + (a_0 + b_0 x) y = 0,$$

where $b_3 = 1$.

Let

$$y = e^{\int x^2 Z dx},$$

where the limits are to be determined. Substituting in the differential equation, and following the usual method, we find

$$Z = \frac{1}{b_3(iz)^3 + b_2(iz)^2 + b_1(iz) + b_0} e^{\int dz \frac{a_3(iz)^3 + a_2(iz)^2 + a_1(iz) + a_0}{b_3(iz)^3 + b_2(iz)^2 + b_1(iz) + b_0}}.$$

Now let

$$\frac{a_3(iz)^3 + a_2(iz)^2 + a_1(iz) + a_0}{b_3(iz)^3 + b_2(iz)^2 + b_1(iz) + b_0} = m - \frac{\lambda}{iz - \alpha} - \frac{\mu}{iz - \beta} - \frac{\nu}{iz - \gamma},$$

then we shall have

$$y = \int \frac{e^{iz(m+x)} \cdot dz}{(iz - \alpha)^{\lambda+1} (iz - \beta)^{\mu+1} (iz - \gamma)^{\nu+1}},$$

where the limits, determined by the usual method, are evidently $-\infty$ and $+\infty$.

We have, from the above equation,

$$a_3 = b_3 m,$$

$$a_2 = -b_3 \{\lambda + \mu + \nu + m(\alpha + \beta + \gamma)\},$$

$$a_1 = b_3 \{\lambda(\beta + \gamma) + \mu(\alpha + \gamma) + \nu(\alpha + \beta) + m(\alpha\beta + \alpha\gamma + \beta\gamma)\},$$

$$a_0 = -b_3 \{\lambda\beta\gamma + \mu\alpha\gamma + \nu\alpha\beta + m\alpha\beta\gamma\},$$

$$b_2 = -b_3(\alpha + \beta + \gamma),$$

$$b_1 = b_3(\alpha\beta + \alpha\gamma + \beta\gamma),$$

$$b_0 = -b_3\alpha\beta\gamma.$$

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Hence the differential equation becomes

$$\begin{aligned} (m+x) \frac{d^2 y}{dx^3} - \{\lambda + \mu + \nu + (\alpha + \beta + \gamma)(m+x)\} \frac{d^2 y}{dx^2} \\ + \{\lambda(\beta + \gamma) + \mu(\alpha + \gamma) + \nu(\alpha + \beta) + (\alpha\beta + \beta\gamma + \alpha\gamma)(m+x)\} \frac{dy}{dx} \\ - \{\lambda\beta\gamma + \mu\alpha\gamma + \nu\alpha\beta + \alpha\beta\gamma(m+x)\} y = 0. \end{aligned}$$

The solution of this equation will be as follows :—

$$y = P e^{\alpha x} + Q e^{\beta x} + R e^{\gamma x},$$

where

$$P = A + Bx + Cx^2 + \dots + Hx^\lambda,$$

$$Q = A' + B'x + C'x^2 + \dots + H'x^\mu,$$

$$R = A'' + B''x + C''x^2 + \dots + H''x^\nu.$$

where $A, B, \dots, A', B', \dots, A'', B'', \dots$ are constants to be determined ;

$$\therefore \int_{-\infty}^{\infty} \frac{e^{iz(m+x)} dz}{(iz-\alpha)^{\lambda+1} (iz-\beta)^{\mu+1} (iz-\gamma)^{\nu+1}} = P e^{\alpha x} + Q e^{\beta x} + R e^{\gamma x}.$$

Now let α be essentially negative, β and γ essentially positive. Then, since the integral cannot always go on increasing with (x) , we have $Q=0$, $R=0$;

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} \frac{e^{iz(m+x)} d \cdot iz}{(iz-\alpha)^{\lambda+1} (iz-\beta)^{\mu+1} (iz-\gamma)^{\nu+1}} \\ = P e^{\alpha x} = (A + Bx + Cx^2 + \dots + Hx^\lambda) e^{\alpha x} : \end{aligned}$$

the constants $A, B, C, \dots$, where one of them is known, may be determined by substitution in the differential equation

$$\begin{aligned} (m+x) \frac{d^3 y}{dx^3} - \{\lambda + \mu + \nu + (\beta + \gamma - 2\alpha)(m+x)\} \frac{d^2 y}{dx^2} \\ - \{\lambda(2\alpha - \beta - \gamma) + \mu(\alpha - \gamma) + \nu(\alpha - \beta) \\ - (\alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma)(m+x)\} \frac{dy}{dx} - \lambda(\alpha^2 - \alpha\beta - \alpha\gamma + \beta\gamma)y = 0. \end{aligned}$$

But they are better determined in the following way. We may evidently, without any loss of generality, put $m=0$. In this case we shall have

$$\int_{-\infty}^{\infty} \frac{e^{ixz} d \cdot iz}{(iz-\alpha)^{\lambda+1} (iz-\beta)^{\mu+1} (iz-\gamma)^{\nu+1}} = (A + Bx + Cx^2 + \dots + Hx^\lambda) e^{\alpha x}.$$

If we put $x=0$ in this integral, we have

$$A = \int_{-\infty}^{\infty} \frac{d \cdot iz}{(iz-\alpha)^{\lambda+1} (iz-\beta)^{\mu+1} (iz-\gamma)^{\nu+1}}.$$

Differentiating and putting $x=0$,

$$\int_{-\infty}^{\infty} \frac{iz \cdot dz}{(iz-\alpha)^{\lambda+1}(iz-\beta)^{\mu+1}(iz-\gamma)^{\nu+1}} = A\alpha + B;$$

$$\therefore B = \int_{-\infty}^{\infty} \frac{d \cdot iz}{(iz-\alpha)^{\lambda}(iz-\beta)^{\mu+1}(iz-\gamma)^{\nu+1}};$$

and so we may proceed. The integral is thus completely determined.

A similar process will give us the more general integral

$$\int_{-\infty}^{\infty} \frac{\epsilon^{ixz} d \cdot iz}{(iz-\alpha)^{\lambda+1}(iz-\beta)^{\mu+1}(iz-\gamma)^{\nu+1} \dots (iz-\zeta)^{\omega+1}},$$

where α is essentially negative, and $\beta, \gamma \dots \zeta$ essentially positive*.

The same process will hold good when $\mu, \nu \dots$ are fractional; for it is manifest that the integral

$$\int_{-\infty}^{\infty} \frac{\epsilon^{ixz} \cdot dz}{(iz-\alpha)^{\lambda+1}(iz-\beta)^{\mu+1}(iz-\gamma)^{\nu+1}} = P\epsilon^{\alpha x} + Q\epsilon^{\beta x} + R\epsilon^{\gamma x};$$

where, however,

$$Q = \dots \frac{A_{\nu}}{x^{\nu-\lambda}} + \frac{A_{\nu-1}}{x^{\nu-\lambda-1}} + \dots + A_2 x^{\lambda-2} + A_1 x^{\lambda-1} + A_0 x^{\lambda},$$

with a similar expression for R .

Now, as x increases without limit, the series

$$\frac{A_{\nu}}{x^{\nu-\lambda}} + \frac{A_{\nu+1}}{x^{\nu-\lambda+1}} + \dots$$

converges; for since x is arbitrary, the ratio of two consecutive terms, $\frac{A_{\nu-\lambda+1}}{A_{\nu}x}$, may be made as small as we please, however great ν becomes. Hence, as x increases without limit, the series $\frac{A_{\nu-\lambda}}{x^{\nu-\lambda}} + \dots$ approaches zero. Therefore, as we suppose β positive, $Q\epsilon^{\beta x}$ will increase without limit; and therefore as Q is supposed multiplied by an arbitrary constant, we must have $Q=0$. Hence the value of the integral where $\mu, \nu, \dots$ are supposed fractional will also be $P\epsilon^{\alpha x}$, α being negative, and the constants in P determined as before.

Next, consider the integral

$$\int_{\alpha}^{\infty} \frac{\epsilon^{zx}(z-\alpha)^{\lambda-1} dz}{(z-\beta)^{\mu+1}(z-\gamma)^{\nu+1}},$$

where x is supposed negative, α and β essentially positive, β less than α ,

* It is proper here to remind the reader that the integral

$$\int_{-\infty}^{\infty} \frac{\epsilon^{i\beta x} dx}{(\alpha + ix)^n} = \frac{2\pi\beta^{m-1}\epsilon^{-\alpha\beta}}{\Gamma(n)}$$

does not hold good when α is negative.

γ essentially negative. Then the integral will satisfy the differential equation

$$\begin{aligned} x \frac{d^3 y}{dx^3} + \{\lambda - \mu - \nu - (\alpha + \beta + \gamma)x\} \frac{d^2 y}{dx^2} \\ - \{\lambda(\beta + \gamma) - \mu(\alpha + \gamma) - \nu(\alpha + \beta) - (\alpha\beta + \beta\gamma + \alpha\gamma)x\} \frac{dy}{dx} \\ - \{\lambda\beta\gamma + \mu\alpha\gamma + \nu\alpha\beta + \alpha\beta\gamma x\} y = 0. \end{aligned}$$

We here suppose λ fractional, μ and ν positive and entire. The solution of the differential equation will be

$$y = P e^{\beta x} + Q e^{\gamma x} + C \{ P e^{\beta x} \int X_1 e^{\frac{\alpha + \beta + \gamma}{2} x^2} dx + Q e^{\gamma x} \int X_2 e^{\frac{\alpha + \beta + \gamma}{2} x^2} dx \};$$

where P and Q are rational and entire functions of the orders μ and ν respectively, and X_1, X_2 functions of x , which it will be needless to write down. The integral will consequently be equal to this expression when the arbitrary constants are properly determined. The integral cannot go on increasing as x , supposed negative, increases. Hence $Q=0$ and $C=0$, and the integral becomes

$$\int_{\alpha}^{\infty} \frac{e^{zx}(z-\alpha)^{\lambda-1} dz}{(z-\beta)^{\mu+1}(z-\gamma)^{\nu+1}} = (A + Bx + Cx^2 + \dots + Hx^{\mu}) e^{\beta x},$$

where $A, B, \&c.$ are constants to be determined.

We find, as before,

$$A = \int_{\alpha}^{\infty} \frac{(z-\alpha)^{\lambda-1} dz}{(z-\beta)^{\mu+1}(z-\gamma)^{\nu+1}},$$

$$B = \int_{\alpha}^{\infty} \frac{(z-\alpha)^{\lambda-1} dz}{(z-\beta)^{\mu}(z-\gamma)^{\nu+1}},$$

$\&c. = \&c.$

And thus the integral is completely known.

I will now consider the differential equation

$$2n \frac{d^3 y}{dx^3} - x \frac{d^2 y}{dx^2} + 2na^2 \frac{dy}{dx} - a^2 xy = 0.$$

It is easily seen that this equation is satisfied by the integral

$$y = \int_{-\infty}^{\infty} \frac{e^{-nu^2 + xu} du}{a^2 + u^2}.$$

The solution of the differential equation is

$$y = C_1 \cos ax + C_2 \sin ax + C_3 \{ \cos ax \int e^{x^2} \sin ax dx - \sin ax \int e^{x^2} \cos ax dx \};$$

and these must be equivalent.

If we put $-u$ for u in this equation, we easily see that

$$\int_{-\infty}^{\infty} \frac{e^{-nu^2 + xu} du}{a^2 + u^2} = \int_{-\infty}^{\infty} \frac{e^{-nu^2 - xu} du}{a^2 + u^2};$$

hence the integral is unchanged if $-x$ is put for x , and therefore $C_2 = C_3 = 0$.

Hence

$$\int_{-\infty}^{\infty} \frac{e^{-nu^2+xu} du}{a^2+u^2} = c \cos ax;$$

$\therefore$ putting $x=0$,

$$c = \int_{-\infty}^{\infty} \frac{e^{-nu^2} du}{a^2+u^2};$$

$$\therefore \frac{dc}{dn} - a^2 c = \int_{-\infty}^{\infty} e^{-nu^2} du = -\frac{\sqrt{\pi}}{\sqrt{n}}.$$

Integrating this equation, and choosing the arbitrary constant so that c may vanish when n is infinite,

$$c = \sqrt{\pi} e^{a^2 n} \int_n^{\infty} \frac{dn}{\sqrt{n}} e^{-a^2 n}.$$

Hence we shall have

$$\int_{-\infty}^{\infty} \frac{e^{-nu^2+xu} du}{a^2+u^2} = 2\sqrt{\pi} \cos ax e^{a^2 n} \int_n^{\infty} \frac{d\mu}{\sqrt{\mu}} e^{-a^2 \mu^2},$$

which last integral is exceedingly well known. It is manifest that we can reduce the integral $\int_{-\infty}^{\infty} \frac{e^{-nu^2+xu} du}{a^2+u^2}$ to this by the method of partial fractions.

In concluding this paper, I desire to express the obligations I am under to Spitzer's 'Studien.'

II. "Measurements of Specific Inductive Capacity of Dielectrics, in the Physical Laboratory of the University of Glasgow." By JOHN C. GIBSON, M.A., and THOMAS BARCLAY, M.A. Communicated by Sir WILLIAM THOMSON. Received November 23, 1870.

(Abstract.)

This paper describes the instruments and processes employed in a series of experiments on the specific inductive capacity of paraffine, and the effect upon it of variations of temperature. The instruments described are the platymeter and the sliding condenser. The former of these was, in a rudimentary form, shown to the Mathematical and Physical Section of the British Association at its Glasgow Meeting in 1855, by W. Thomson. It consists of two equal and similar condensers employed for the comparison of electrostatic capacities. The sliding condenser is a condenser the capacity of which may be varied by known quantities by altering the effective area of the opposed surfaces. By means of these two instruments, along with the quadrant electrometer, the capacity of a condenser may be determined by equalizing the sliding condenser to it. The method of working, and the electrical actions upon which it depends, are described in detail. In